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Oscillation of certain higher-order neutral partial functional differential equations

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Department of Mathematics,
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Republic of China**Abstract**

In this paper, we study the oscillation of certain higher-order neutral partial functional differential equations with the Robin boundary conditions. Some oscillation criteria are established. Two examples are given to illustrate the main results in the end of this paper.

Keywords: Oscillation, Partial functional differential equation, Robin boundary condition

Mathematics Subject Classification: 35B05, 35R10

Background

It is well known that the theory of partial functional differential equations can be applied to many fields, such as population dynamics, cellular biology, meteorology, viscoelasticity, engineering, control theory, physics and chemistry (Wu 1996). In the monograph, Wu (1996) provided some fundamental theories and applications of partial functional differential equations.

The oscillation theory as a part of the qualitative theory of partial functional differential equations has been developed in the past few years. Many researchers have established some oscillation results for partial functional differential equations. For example, see the monograph (Yoshida 2008) and the papers (Bainov et al. 1996; Fu and Zhuang 1995; Li and Cui 1999; Li 2000; Li and Cui 2001; Ouyang et al. 2005; Gao and Luo 2008; Li and Han 2006; Wang et al. 2010). We especially note that the monograph (Yoshida 2008) contained large material on oscillation theory for partial differential equations.

Li and Cui (2001) studied the oscillation of even order partial functional differential equations

$$\begin{aligned} \frac{\partial^n}{\partial t^n} [u(x, t) + \mu(t)u(x, t - \rho)] &= a(t)\Delta u(x, t) + \sum_{k=1}^s a_k(t)\Delta u(x, \rho_k(t)) \\ &\quad - q(x, t)u(x, t) - \int_a^b p(x, t, \xi)u(x, g(t, \xi))d\sigma(\xi), \quad (x, t) \in \Omega \times [0, \infty) \equiv G, \end{aligned} \quad (E1)$$

where $n \geq 2$ is an even integer, with the two kinds of boundary conditions:

$$\frac{\partial u(x, t)}{\partial N} + v(x, t)u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, \infty), \quad (\text{B1})$$

and

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, \infty). \quad (\text{B2})$$

Ouyang et al. (2005) established the oscillation of odd order partial functional differential equations

$$\begin{aligned} \frac{\partial^n u(x, t)}{\partial t^n} - a(t)\Delta u(x, t) - \sum_{k=1}^s p_k(x, t)u(x, t - \sigma_k) - \sum_{j=1}^m q_j(x, t)u(x, t - \tau_j) \\ + h(t)f(u(x, t - r_1), \dots, u(x, t - r_\ell)) = 0, \quad (x, t) \in \Omega \times [0, \infty) \equiv G, \end{aligned} \quad (\text{E2})$$

where n is an odd integer and $s \leq m$, with the boundary conditions (B1), (B2) and

$$\frac{\partial u(x, t)}{\partial N} = 0, \quad (x, t) \in \partial\Omega \times [0, \infty). \quad (\text{B3})$$

In this paper, we investigate the oscillation of the following higher-order neutral partial functional differential equations

$$\begin{aligned} \frac{\partial^n}{\partial t^n} [u(x, t) + \mu(t)u(x, t - \tau)] = a(t)\Delta u(x, t) + \sum_{k=1}^s a_k(t)\Delta u(x, \rho_k(t)) \\ - \int_a^b p(t, \xi)u(x, g(t, \xi))d\sigma(\xi), \quad (x, t) \in \Omega \times [0, \infty) \equiv G, \end{aligned} \quad (1)$$

with the Robin boundary condition

$$\alpha(x)\frac{\partial u(x, t)}{\partial N} + \beta(x)u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, \infty), \quad (2)$$

where $n \geq 2$ is an even integer, Ω is a bounded domain in \mathbb{R}^M with a piecewise smooth boundary $\partial\Omega$, and Δ is the Laplacian in the Euclidean M -space \mathbb{R}^M , $\alpha, \beta \in C(\partial\Omega, [0, \infty))$, $\alpha^2(x) + \beta^2(x) \neq 0$, and N is the unite exterior normal vector to $\partial\Omega$.

Throughout this paper, we always suppose that the following conditions hold:

- (C1) $\mu \in C^n([0, \infty); [0, \infty))$, $0 \leq \mu(t) \leq 1$, $\tau = \text{const.} > 0$;
- (C2) $a, a_k \in C([0, \infty); [0, \infty))$, $\rho_k \in C([0, \infty); [0, \infty))$, $\rho_k(t) \leq t$, $\lim_{t \rightarrow +\infty} \rho_k(t) = +\infty$, $k \in I_s = \{1, 2, \dots, s\}$;
- (C3) $p \in C([0, \infty) \times [a, b]; [0, \infty))$, $g \in C([0, \infty) \times [a, b]; [0, \infty))$, $g(t, \xi) \leq t$, $\xi \in [a, b]$, $g(t, \xi)$ is a nondecreasing function with respect to t and ξ , respectively, and $\lim_{t \rightarrow +\infty} \inf_{\xi \in [a, b]} \{g(t, \xi)\} = +\infty$;
- (C4) $\sigma \in ([a, b]; \mathbb{R})$ and $\sigma(\xi)$ is nondecreasing in ξ , the integral in (1) is Stieltjes integral.

As it is customary, the solution $u(x, t) \in C^n(G) \cap C^1(\overline{G})$ of the problem (1), (2) is said to be oscillatory in the domain $G \equiv \Omega \times [0, \infty)$ if for any positive number μ there exists a point $(x_0, t_0) \in \Omega \times [\mu, \infty)$ such that the equality $u(x_0, t_0) = 0$ holds.

To the best of our knowledge, no result is known regarding the oscillatory behavior of higher-order partial functional differential equations with the Robin boundary condition (2) up to now.

The paper is organized as follows. In “Main results” section, we establish some results for the oscillation of the problem (1), (2). In “Examples” section, we construct two examples to illustrate our main results.

Main results

In this section, we establish the oscillation criteria of the problem (1), (2). First, we introduce the following lemma which is very useful for establishing our main results.

Lemma 1 Ye and Li (1990). Suppose that λ_0 is the smallest eigenvalue of the problem

$$\begin{cases} \Delta \varphi(x) + \lambda \varphi(x) = 0, & \text{in } \Omega, \\ \alpha(x) \frac{\partial \varphi(x)}{\partial N} + \beta(x) \varphi(x) = 0, & \text{on } \partial \Omega \end{cases} \quad (3)$$

and $\varphi(x)$ is the corresponding eigenfunction of λ_0 . Then $\lambda_0 = 0, \varphi(x) = 1$ as $\beta(x) = 0$ ($x \in \Omega$) and $\lambda_0 > 0, \varphi(x) > 0$ ($x \in \Omega$) as $\beta(x) \neq 0$ ($x \in \partial \Omega$).

Next, we give our main results.

Theorem 2 If $\beta(x) \neq 0$ for $x \in \partial \Omega$, then the necessary and sufficient condition for all solutions of the problem (1), (2) to oscillate is that all solutions of the differential equation

$$\begin{aligned} & [y(t) + \mu(t)y(t - \tau)]^{(n)} + \lambda_0 a(t)y(t) + \lambda_0 \sum_{k=1}^s a_k(t)y(\rho_k(t)) \\ & + \int_a^b p(t, \xi)y(g(t, \xi))d\sigma(\xi) = 0, \quad t \geq 0, \end{aligned} \quad (4)$$

to oscillate, where λ_0 is the smallest eigenvalue of (3).

Proof (i) Sufficiency. Suppose to the contrary that there is a non-oscillatory solution $u(x, t)$ of the problem (1), (2) which has no zero in $\Omega \times [t_0, \infty)$ for some $t_0 \geq 0$. Without loss of generality we assume that $u(x, t) > 0, u(x, t - \tau) > 0, u(x, \rho_k(t)) > 0, u(x, g(t, \xi)) > 0, (x, t) \in \Omega \times [t_1, \infty), k \in I_s$.

Multiplying both sides of (1) by $\varphi(x)$ and integrating with respect to x over the domain Ω , we have

$$\begin{aligned} & \frac{d^n}{dt^n} \left[\int_{\Omega} u(x, t) \varphi(x) dx + \mu(t) \int_{\Omega} u(x, t - \tau) \varphi(x) dx \right] \\ & = a(t) \int_{\Omega} \Delta u(x, t) \varphi(x) dx + \sum_{k=1}^s a_k(t) \int_{\Omega} \Delta u(x, \rho_k(t)) \varphi(x) dx \\ & \quad - \int_{\Omega} \int_a^b p(t, \xi) u(x, g(t, \xi)) \varphi(x) d\sigma(\xi) dx, \quad t \geq t_1. \end{aligned} \quad (5)$$

From Green's formula and boundary condition (2), it follows that

$$\begin{aligned} & \int_{\Omega} \Delta u(x, t) \varphi(x) dx \\ &= \int_{\partial\Omega} \left(\varphi(x) \frac{\partial u(x, t)}{\partial N} - u(x, t) \frac{\partial \varphi(x)}{\partial N} \right) dS + \int_{\Omega} u(x, t) \Delta \varphi(x) dx \\ &= \int_{\partial\Omega} \left(\varphi(x) \frac{\partial u(x, t)}{\partial N} - u(x, t) \frac{\partial \varphi(x)}{\partial N} \right) dS - \lambda_0 \int_{\Omega} u(x, t) \varphi(x) dx, \quad t \geq t_1, \end{aligned}$$

where dS is the surface element on $\partial\Omega$.

If $\alpha(x) \equiv 0, x \in \partial\Omega$, then from (2) we have

$$\beta(x) \neq 0, \quad u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, \infty).$$

Hence, we obtain

$$\int_{\partial\Omega} \left(\varphi(x) \frac{\partial u(x, t)}{\partial N} - u(x, t) \frac{\partial \varphi(x)}{\partial N} \right) dS \equiv 0, \quad t \geq t_1.$$

If $\alpha(x) \neq 0, x \in \partial\Omega$. Noting that $\partial\Omega$ is piecewise smooth, $\alpha, \beta \in C(\partial\Omega, [0, \infty))$, $\alpha^2(x) + \beta^2(x) \neq 0$, without loss of generality, we can assume that $\alpha(x) > 0, x \in \partial\Omega$. Then by (2) and (3) we have

$$\begin{aligned} & \int_{\partial\Omega} \left(\varphi(x) \frac{\partial u(x, t)}{\partial N} - u(x, t) \frac{\partial \varphi(x)}{\partial N} \right) dS \\ &= \int_{\partial\Omega} \left(-\varphi(x) \frac{\beta(x)}{\alpha(x)} u(x, t) + u(x, t) \frac{\beta(x)}{\alpha(x)} \varphi(x) \right) dS = 0, \quad t \geq t_1, \end{aligned}$$

Therefore, using Lemma 1, we obtain

$$\int_{\Omega} \Delta u(x, t) \varphi(x) dx = -\lambda_0 \int_{\Omega} u(x, t) \varphi(x) dx, \quad t \geq t_1. \quad (6)$$

Similarly, we have

$$\int_{\Omega} \Delta u(x, \rho_k(t)) \varphi(x) dx = -\lambda_0 \int_{\Omega} u(x, \rho_k(t)) \varphi(x) dx, \quad t \geq t_1, \quad k \in I_s. \quad (7)$$

It is easy to see that

$$\begin{aligned} & \int_{\Omega} \int_a^b p(t, \xi) u(x, g(t, \xi)) \varphi(x) d\sigma(\xi) dx \\ &= \int_a^b p(t, \xi) \int_{\Omega} u(x, g(t, \xi)) \varphi(x) dx d\sigma(\xi), \quad t \geq t_1. \end{aligned} \quad (8)$$

Set

$$V(t) = \int_{\Omega} u(x, t) \varphi(x) dx, \quad t \geq t_1.$$

Combining (5)–(8) we have

$$\begin{aligned} & [V(t) + \mu(t)V(t - \tau)]^{(n)} + \lambda_0 a(t)V(t) + \lambda_0 \sum_{k=1}^s a_k(t)V(\rho_k(t)) \\ & + \int_a^b p(t, \xi)V(g(t, \xi))d\sigma(\xi) = 0, \quad t \geq t_1. \end{aligned} \quad (9)$$

Obviously, it follows from (9) that $V(t)$ is a positive solution of Eq. (4), which contradicts the fact that all solutions of Eq. (4) are oscillatory.

(ii) Necessity. Suppose that Eq. (4) has a non-oscillatory solution $\tilde{V}(t) > 0$. Without loss of generality we assume $\tilde{V}(t) > 0$ for $t \geq t_* \geq 0$, where t_* is some large number. From (4), we have

$$\begin{aligned} & [\tilde{V}(t) + \mu(t)\tilde{V}(t - \tau)]^{(n)} + \lambda_0 a(t)\tilde{V}(t) + \lambda_0 \sum_{k=1}^s a_k(t)\tilde{V}(\rho_k(t)) \\ & + \int_a^b p(t, \xi)\tilde{V}(g(t, \xi))d\sigma(\xi) = 0, \quad t \geq t_*. \end{aligned} \quad (10)$$

Multiplying both sides of (10) by $\varphi(x)$, we obtain

$$\begin{aligned} & \frac{\partial^n}{\partial t^n} \left[\tilde{V}(t)\varphi(x) + \mu(t)\tilde{V}(t - \tau)\varphi(x) \right] \\ & + \lambda_0 a(t)\tilde{V}(t)\varphi(x) + \lambda_0 \sum_{k=1}^s a_k(t)\tilde{V}(\rho_k(t))\varphi(x) \\ & + \int_a^b p(t, \xi)\tilde{V}(g(t, \xi))\varphi(x)d\sigma(\xi) = 0, \quad t \geq t_*, \quad x \in \Omega. \end{aligned} \quad (11)$$

Let $\tilde{u}(x, t) = \tilde{V}(t)\varphi(x)$, $(x, t) \in \Omega \times [0, \infty)$. By Lemma 1, we have $\Delta\varphi(x) = -\lambda_0\varphi(x)$, $x \in \Omega$. Then (11) implies

$$\begin{aligned} & \frac{\partial^n}{\partial t^n} \left[\tilde{u}(x, t) + \mu(t)\tilde{u}(x, t - \tau) \right] = a(t)\Delta\tilde{u}(x, t) + \sum_{k=1}^s a_k(t)\Delta\tilde{u}(x, \rho_k(t)) \\ & - \int_a^b p(t, \xi)\tilde{u}(x, g(t, \xi))d\sigma(\xi), \quad t \geq t_*, \quad x \in \Omega, \end{aligned} \quad (12)$$

which shows that $\tilde{u}(x, t) = \tilde{V}(t)\varphi(x)$, $(x, t) \in \Omega \times [t_*, \infty)$, satisfies (1).

From Lemma 1, we get

$$\alpha(x)\frac{\partial\varphi(x)}{\partial N} + \beta(x)\varphi(x) = 0, \quad x \in \partial\Omega,$$

which implies

$$\alpha(x)\frac{\partial\tilde{u}(x, t)}{\partial N} + \beta(x)\tilde{u}(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, \infty). \quad (13)$$

Hence $\tilde{u}(x, t) = \tilde{V}(t)\varphi(x) > 0$ is a non-oscillatory solution of the problem (1), (2), which is a contradiction. The proof is complete. \square

Remark 3 Theorem 2 shows that the oscillation of problem (1), (2) is equivalent to the oscillation of the differential equation (4).

Theorem 4 If $\beta(x) \not\equiv 0$ for $x \in \partial\Omega$, and the neutral differential inequality

$$[y(t) + \mu(t)y(t - \tau)]^{(n)} + \int_a^b p(t, \xi)y(g(t, \xi))d\sigma(\xi) \leq 0, \quad t \geq 0, \quad (14)$$

has no eventually positive solutions, then every solution of the problem (1), (2) is oscillatory in G .

Proof Suppose to the contrary that there is a non-oscillatory solution $u(x, t)$ of the problem (1), (2) which has no zero in $\Omega \times [t_0, \infty)$ for some $t_0 \geq 0$. Without loss of generality we assume that $u(x, t) > 0$, $u(x, t - \tau) > 0$, $u(x, \rho_k(t)) > 0$, $u(x, g(t, \xi)) > 0$, $(x, t) \in \Omega \times [t_1, \infty)$, $k \in I_s$. As in the proof of Theorem 2, we obtain Eq. (9). By Lemma 1, from (9) we have

$$\begin{aligned} & [V(t) + \mu(t)V(t - \tau)]^{(n)} + \int_a^b p(t, \xi)V(g(t, \xi))d\sigma(\xi) \\ &= -\lambda_0 a(t)V(t) - \lambda_0 \sum_{k=1}^s a_k(t)V(\rho_k(t)) \\ &\leq 0, \quad t \geq t_1, \end{aligned} \quad (15)$$

which shows that $V(t) > 0$ is a solution of the inequality (14). This is a contradiction. The proof of Theorem 4 is complete. \square

Using Theorems 1 and 2 in Li and Cui (2001), we can obtain the following two conclusions, respectively.

Theorem 5 Assume that $\beta(x) \not\equiv 0$ for $x \in \partial\Omega$. If for $t_0 > 0$,

$$\int_{t_0}^{+\infty} \int_a^b p(s, \xi)[1 - \mu(g(s, \xi))]d\sigma(\xi)ds = +\infty, \quad (16)$$

then every solution of the problem (1), (2) is oscillatory in G .

Theorem 6 Assume that $\beta(x) \not\equiv 0$ for $x \in \partial\Omega$, $\mu(t) \equiv \mu$ is a positive constant, $p(t, \xi)$ is periodic in t with period ρ . If for $t_0 > 0$,

$$g(t - c, \xi) = g(t, \xi) - c \quad \text{for any number } c > 0, \quad (17)$$

$$\int_{t_0}^{+\infty} \int_a^b p(s, \xi)d\sigma(\xi)ds = +\infty, \quad (18)$$

then every solution of the problem (1), (2) is oscillatory in G .

Examples

In this section, we give two examples to illustrate our main results.

Example 7 Consider the partial functional differential equation

$$\begin{aligned} \frac{\partial^6}{\partial t^6} \left[u(x, t) + \frac{1}{5} u(x, t - \pi) \right] &= 3 \Delta u(x, t) + \frac{11}{5} \Delta u \left(x, t - \frac{3\pi}{2} \right) \\ &- \int_{-\pi}^{-\pi/2} \frac{11}{5} u(x, t + \xi) d\xi, \quad (x, t) \in (0, \pi) \times [0, \infty), \end{aligned} \quad (19)$$

with the boundary condition

$$u(0, t) = u(\pi, t) = 0, \quad t \geq 0. \quad (20)$$

Here $n = 6, \mu(t) = \frac{1}{5}, \tau = \pi, a(t) = 3, a_1(t) = \frac{11}{5}, \rho_1(t) = t - \frac{3\pi}{2}, p(t, \xi) = \frac{11}{5}, g(t, \xi) = t + \xi, \sigma(\xi) = \xi, a = -\pi, b = -\frac{\pi}{2}$. It is easy to see that for $t_0 > 0$,

$$\int_{t_0}^{+\infty} \int_a^b p(s, \xi) [1 - \mu(g(s, \xi))] d\sigma(\xi) ds = \int_{t_0}^{+\infty} \int_{-\pi}^{-\pi/2} \frac{11}{5} \left[1 - \frac{1}{5} \right] d\xi ds = +\infty.$$

Then the conditions of Theorem 5 are fulfilled. Therefore every solution of the problem (19), (20) is oscillatory in $(0, \pi) \times [0, \infty)$. Indeed, $u(x, t) = \sin x \cos t$ is such a solution.

Example 8 Consider the partial functional differential equation

$$\begin{aligned} \frac{\partial^4}{\partial t^4} \left[u(x, t) + \frac{1}{2} u(x, t - \pi) \right] &= \frac{1}{3} \Delta u(x, t) + \frac{1}{6} \Delta u \left(x, t - \frac{\pi}{2} \right) \\ &- \int_{-\pi}^{-\pi/2} \frac{1}{6} u(x, t + \xi) d\xi, \quad (x, t) \in (0, \pi) \times [0, \infty), \end{aligned} \quad (21)$$

with the boundary condition

$$u_x(0, t) + u(0, t) = u_x(\pi, t) + u(\pi, t) = 0, \quad t \geq 0. \quad (22)$$

Here $n = 4, \mu(t) = \frac{1}{2}, \tau = \pi, a(t) = \frac{1}{3}, a_1(t) = \frac{1}{6}, \rho_1(t) = t - \frac{\pi}{2}, p(t, \xi) = \frac{1}{6}, g(t, \xi) = t + \xi, \sigma(\xi) = \xi, a = -\pi, b = -\frac{\pi}{2}$. It is easy to see that for $t_0 > 0$,

$$\int_{t_0}^{+\infty} \int_a^b p(s, \xi) [1 - \mu(g(s, \xi))] d\sigma(\xi) ds = \int_{t_0}^{+\infty} \int_{-\pi}^{-\pi/2} \frac{1}{6} \left[1 - \frac{1}{2} \right] d\xi ds = +\infty,$$

which shows that the conditions of Theorem 5 are satisfied. By Theorem 5, we obtain that every solution of the problem (21), (22) is oscillatory in $(0, \pi) \times [0, \infty)$. In fact, $u(x, t) = e^{-x} \cos t$ is such a solution.

Conclusions

This paper provides some oscillation criteria for solutions of higher-order neutral partial functional differential equations with Robin boundary conditions. Using Lemma 1, we obtain Theorems 2 and 4. We should note that Theorem 2 shows that the oscillation of

the problem (1), (2) is equivalent to the oscillation of the functional differential equation (4). Using the results in Li and Cui (2001), two useful conclusions are established in Theorems 5 and 6.

Authors' contributions

Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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Competing interests

Both authors declare that they have no competing interests.

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